EX: $A = \begin{bmatrix} -4 & 2 & -2 \\ 2 & -7 & 4 \\ -2 & 4 & -7 \end{bmatrix}$

Sol 1° $p(\lambda) = \begin{vmatrix} \lambda+4 & -2 & 2 \\ -2 & \lambda+7 & -4 \\ 2 & -4 & \lambda+7 \end{vmatrix} = \begin{vmatrix} \lambda+4 & -2 & 0 \\ -2 & \lambda+7 & \lambda+3 \\ 2 & -4 & \lambda+3 \end{vmatrix}$

$= (\lambda+4) \begin{vmatrix} \lambda+7 & \lambda+3 \\ -4 & \lambda+3 \end{vmatrix} + 2 \begin{vmatrix} -2 & \lambda+3 \\ 2 & \lambda+3 \end{vmatrix}$

$= (\lambda+3)\left[(\lambda+4)(\lambda+11) - 8\right] = (\lambda+3)^2(\lambda+12)$

2° $\lambda_1 = -3$

$(\lambda_1 - A) = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 4 & -4 \\ 2 & -4 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$v_{11} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \qquad v_{12} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

3° $\lambda_2 = -12$

$\lambda_2 - A = \begin{bmatrix} -8 & -2 & 2 \\ -2 & -5 & -4 \\ 2 & -4 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$ or $\begin{bmatrix} \frac{1}{2} \\ -1 \\ 1 \end{bmatrix}$

4° Use Gram Schmidt to obtain an orthonormal basis for span$\{v_{11}, v_{12}\}$

$u_1 = v_{11}$

$u_2 = v_{12} - \frac{(v_{12}, u_1)}{\|u_1\|^2} u_1 = v_{12} + \frac{4}{5} u_1 = \left[-\frac{2}{5}, \frac{4}{5}, 1\right]^T$

$w_1 = \frac{u_1}{\|u_1\|} = \left[\frac{2\sqrt{5}}{5}, \frac{\sqrt{5}}{5}, 0\right]^T \quad w_2 = \frac{u_2}{\|u_2\|} = \left[-\frac{2\sqrt{5}}{15}, \frac{4\sqrt{5}}{15}, \frac{\sqrt{5}}{3}\right]^T$

$w_3 = v_2/\|v_2\| = \left[\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right]^T$

So $D = \begin{bmatrix} -3 & & \\ & -3 & \\ & & -12 \end{bmatrix} \qquad P = \begin{bmatrix} 2\sqrt{5}/5 & -2\sqrt{5}/15 & 1/3 \\ \sqrt{5}/5 & 4\sqrt{5}/15 & -2/3 \\ 0 & \sqrt{5}/3 & 2/3 \end{bmatrix}$

RMK: We don't apply Gram-Schmidt to v_2 (Consider why?)